A mathematical model for measurements

Akitaka Kishimoto*

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Abstract

We will give a new model for measurements of a quantum system, expressed as the compact operators on a Hilbert space, such that the measuring apparatuses are described by a unital separable non-type I nuclear simple C*-algebra equipped with certain unital endomorphisms and pure states. An interaction between the quantum system and the apparatus is specified by a unitary associated with the combined system as before. Magnifying to the classical level some aspects of the quantum system so captured in the apparatus is explicitly done by applying the endomorphism; then the resulting state is the superposition of *phases* with weights. Nature will then choose each phase according to the probability prescribed by the weights just as does one when multiple phases appear as in phase transition. Thus in our model state-reduction (or collapse of the wave function) is a primary event; whether this corresponds to the measurement of an observable or which one if it does is another matter.

1 Introduction and Results

In the standard setting originated from von Neumann's book [17], a *measuring process* for the quantum system on a Hilbert space $\mathcal{H}$ is described by a quadruple $(\mathcal{L}, \phi, M, U)$, where $\mathcal{L}$ is a separable Hilbert space, ϕ is a state[1] of the compact operators $\mathcal{K}(\mathcal{L})$, M is a self-adjoint operator on $\mathcal{L}$, and U is a unitary on $\mathcal{H} \otimes \mathcal{L}$.[2] Let E_M denote the spectral measure of M and let $\mathcal{K}(\mathcal{H})^*_+$ denote the cone[3] of positive functionals on $\mathcal{K}(\mathcal{H})$ and $\mathcal{S}(\mathcal{K}) = \{\varphi \in \mathcal{K}(\mathcal{H})^*_+ \mid \|\varphi\| = 1\}$, the convex state space of $\mathcal{K}$. This process produces an $\mathcal{E}(\Delta, \varphi) \in \mathcal{K}^*_+$ for each Borel subset Δ of $\mathbb{R}$ and $\varphi \in \mathcal{S}(\mathcal{K})$ by

$$\mathcal{E}(\Delta, \varphi)(x) = \varphi \otimes \phi(U^*(x \otimes E_M(\Delta))U), \quad x \in \mathcal{K}.$$

*E-mail: aki.ksmt@jcom.home.ne.jp; Retired from Hokkaido University

[1] A state ϕ of $\mathcal{K}(\mathcal{L})$ corresponds to a positive trace class operator ρ on $\mathcal{L}$ of trace one; $\phi(x) = \mathrm{Tr}(x\rho),\ x \in \mathcal{K}(\mathcal{L})$.

[2] Taken from [13] with some notational modifications.

[3] This is the same as the cone of positive trace class operators on $\mathcal{H}$.

Note that $\Delta \mapsto \mathcal{E}(\Delta, \varphi)$ is a $\mathcal{K}_+^*$-valued measure on $\mathbb{R}$ with $\mathcal{E}(\mathbb{R}, \varphi) \in \mathcal{S}(\mathcal{K})$ and that $\varphi \mapsto \mathcal{E}(\Delta, \varphi)$ is a continuous affine map from $\mathcal{S}(\mathcal{K})$ into $\mathcal{K}_+^*$ (in norm). Note also that if (Δ_n) is an increasing sequence of Borel subsets of $\mathbb{R}$ then $\mathcal{E}(\Delta_n, \varphi)$ converges to $\mathcal{E}(\bigcup_n \Delta_n, \varphi)$. All what we get from the quadruple $(\mathcal{L}, \phi, M, U)$ with U a unitary on $\mathcal{H} \otimes \mathcal{L}$ is the collection $\mathcal{E}(\Delta, \varphi)$, which is called a *Davies-Lewis instrument* or DL-instrument for short [4]. The self-adjoint operator M is called a *meter observable* in [13] and a *probe observable* in [12]; When the observed system $\mathcal{K}(\mathcal{H})$ is under the state φ and is applied this measuring process, we are supposed to be able to perform a *precise local* measurement of M to get a real number x whose distribution is given by $\Delta \mapsto \mathcal{E}(\Delta, \varphi)(1)$ such that the ensemble of all states of $\mathcal{K}(\mathcal{H})$ after observing $x \in \Delta$ is given by $\mathcal{E}(\Delta, \varphi)/\mathcal{E}(\Delta, \varphi)(1)$ for each Δ (cf. Section 3 of [10]). Given a self-adjoint operator Q on $\mathcal{H}$, observing Q is supposed to be choosing a suitable $(\mathcal{L}, \phi, M, U)$ and applying the above process.

This seems to be all well-established except for how to drive the quantum effect on M to the macroscopic level for observation (cf. [11]). Recently Harada and Ojima describe such an amplification process as well as the preceding interaction between the observed and the probe systems in terms of certain abelian groups by noting a specific property of *regular representation* (section 3 of [9]). Here we propose another mathematical model for measurements of a quantum system in a C*-algebra setting, which incorporates a mechanism of *magnifying quantum effects to the classical level*[4] into the measuring apparatus. In this scheme the state of the quantum system transforms to new ones according to a certain probability law just like the phase does to a new one in phase transition we encounter in equilibrium quantum statistical mechanics.

The (microscopic) quantum system is described by the C*-algebra $\mathcal{K} = \mathcal{K}(\mathcal{H})$ of compact operators on a separable Hilbert space $\mathcal{H}$ just as above and the (macroscopic) measuring apparatus is by a unital separable non-type I nuclear simple C*-algebra A with a certain unital endomorphism and a pure state.[5] We will then specify a unitary from the combined system $\mathcal{K} \otimes A$ to dictate an interaction. After applying the adjoint action of the unitary and the endomorphism we reach a situation similar to the above; instead of M (or the von Neumann algebra generated by M) we will obtain an abelian von Neumann algebra with zero intersection with the compact operators, as the center of the observable algebra, as an outcome of this process.

Let us explain some basic of states and how the endomorphism works. When ϕ is a state of A, i.e., a positive linear functional with $\phi(1) = 1 = \|\phi\|$, we obtain the so-called GNS representation π_ϕ of A on a Hilbert space $\mathcal{H}_\phi$ with a specific unit vector Ω_ϕ such that $\phi(x) = \langle \Omega_\phi, \pi_\phi(x)\Omega_\phi \rangle,\ x \in A$ and $\pi_\phi(A)\Omega_\phi$ is dense in $\mathcal{H}_\phi$. If ϕ is a pure state, i.e., an extreme point of the convex set of states of A, the weak closure of $\pi_\phi(A)$ is all the bounded operators $\mathcal{B}(\mathcal{H}_\phi)$. We call ϕ (or π_ϕ) *factorial* if the weak closure $\mathcal{M}$ of $\pi_\phi(A)$ is a factor, i.e., the intersection of the commutant of $\mathcal{M}$, $\mathcal{M}' = \{Q \in \mathcal{B}(\mathcal{H}_\phi) \mid QT = TQ, T \in \mathcal{M}\}$,

[4] This expression is taken from page 250 of Penrose's book [14].

[5] That the C*-algebra A is non-type I and nuclear is assumed to assure that A has a desired endomorphism [8]. We may assume that A is the UHF algebra of type 2^∞, or the infinite tensor product of 2×2 matrices, which may be considered as the observable algebra for an infinite number of electrons.

with $\mathcal{M}$ is $\mathbb{C}1$. In particular a pure state is factorial. If γ is a unital endomorphism of A with $\gamma(A) \neq A$ and ϕ is a factorial state then $\phi\gamma$ is a state but may not be factorial, i.e., if $\mathcal{M}$ denotes the weak closure of $\pi_{\phi\gamma}(A)$, the center $\mathcal{M} \cap \mathcal{M}'$ of $\mathcal{M}$ may not be $\mathbb{C}1$. (If $\phi\gamma$ is factorial $\pi_\phi\gamma$ may not be factorial, i.e., the weak closure of $\pi_\phi\gamma(A)$ may have non-trivial center. This is because $\pi_{\phi\gamma}$ is the restriction of $\pi_\phi\gamma$ to the subspace defined as the closure of $\pi_\phi\gamma(A)\Omega_\phi$.) In this case $\phi\gamma$ is centrally decomposed in the sense that there is a probability measure ν on the Borel subset $\mathcal{F}$ of factorial states in A^* with

$$\phi\gamma = \int_{\mathcal{F}} \psi \mathrm{d}\nu(\psi),$$

where $\mathcal{M} \cap \mathcal{M}'$ on the left could be identified with $L^\infty(\mathcal{F}, \nu)$ on the right behind this equality (see 3.1.8 and 3.4.5 of [15]). A factorial state is supposed to correspond to a *phase* in statistical mechanics and if ϕ transforms to $\phi\gamma$ causally but in a irreversible way then it would immediately jump to $\psi \in \mathcal{F}$ acausally according to the probability ν on $\mathcal{F}$.[6]

We also assume that γ is asymptotically inner.[7] Namely we assume that there is a continuous unitary path u_t, $t \in [0, 1)$ in A such that $\gamma(x) = \lim_{t\to1} u_t x u_t^*$, $x \in A$ and $u_0 = 1$. Then it follows that there is a bounded sequence (h_n) of self-adjoint elements of A such that $\lim_n [h_n, x] = 0$ and $\gamma(x) = \lim_n \mathrm{Ad}(e^{ih_1} e^{ih_2} \cdots e^{ih_n})(x)$ for $x \in A$. We regard γ as a limit of time developments which are cascading quantum effects to the visible classical level within a small time interval. Thus γ describes an irreversible process.

If φ is a state of $\mathcal{K}(\mathcal{H})$ and the measuring apparatus A is in a pure state ϕ, then we suppose that $\varphi \otimes \phi$ turns to $(\varphi \otimes \phi)\mathrm{Ad}\, U^*$ and then to $(\varphi \otimes \phi)\mathrm{Ad}\, U^*(\mathrm{id} \otimes \gamma)$, which may not be factorial and then will be centrally decomposed as explained above, or we will witness collapse of the wave function.

We formally give the definition of CP instrument and then the definition of our measuring processes (cf. [4, 10, 13]).

Definition 1.1 *Let $\mathcal{M}$ be an abelian von Neumann algebra with separable predual*[8] *and $\mathcal{M}_+$ the cone of positive elements of $\mathcal{M}$. Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space and $\mathcal{K} = \mathcal{K}(\mathcal{H})$ be the C^*-algebra of compact operators on $\mathcal{H}$. We call a map $\mathcal{E}$ from $\mathcal{M} \times \mathcal{K}^*$ into $\mathcal{K}^*$ a* CP instrument *based on $\mathcal{M}$ if it satisfies*

1. *For each $\varphi \in \mathcal{K}^*_+$ the map $\mathcal{M} \ni Q \mapsto \mathcal{E}(Q, \varphi) \in \mathcal{K}^*$ is a positive continuous linear map with $\mathcal{M}$,*

2. *For each $Q \in \mathcal{M}_+$ the map $\mathcal{K}^* \ni \varphi \mapsto \mathcal{E}(Q, \varphi) \in \mathcal{K}^*$ is a completely positive linear map,*

[6]Phases or sectors are also discussed in [9]. See [3] for backgrounds.

[7]This is a misnomer but is widely used among operator algebraists. It is more like being asymptotically NOT inner and means that γ is asymptotically approximated by inner automorphisms.

[8]A Banach space X is a predual of $\mathcal{M}$ if $X^* \cong \mathcal{M}$; $\mathcal{M}$ has a unique predual denoted by $\mathcal{M}_*$ (1.13.3 of [15]). We know that $\mathcal{M}$ is isomorphic to L^∞-space on some probability space.

3. $\mathcal{E}(1,\varphi)(1) = \varphi(1)$ *for all* $\varphi \in \mathcal{K}^*_+$,

where $\mathcal{M}$ *is equipped with the weak***-topology.*

If we denote by $\mathcal{E}(Q)$ the linear map $\mathcal{K}^* \ni \varphi \mapsto \mathcal{E}(Q,\varphi) \in \mathcal{K}^*$ with $Q \in \mathcal{M}_+$, the dual map $\mathcal{E}(Q)^* : \mathcal{K}^{**} = \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is completely positive or CP, i.e., the natural extension of $\mathcal{E}(Q)^*$ to a map from $M_k \otimes \mathcal{B}(\mathcal{H})$ into $M_k \otimes \mathcal{B}(\mathcal{H})$ is positive for any $k \in \mathbb{N}$, which follows from the complete positivity of $\mathcal{E}(Q)$. We denote $\mathcal{E}(Q)^* b$ by $\mathcal{E}^*(Q,b)$ for $b \in \mathcal{B}(\mathcal{H})$; then for each $b \in \mathcal{B}(\mathcal{H})_+$ the map $Q \mapsto \mathcal{E}^*(Q,b)$ is a positive continuous linear map[9] when $\mathcal{M}$ and $\mathcal{B}(\mathcal{H})$ are endowed with the weak*-topology. For $Q \in \mathcal{M}_+$ the map $b \mapsto \mathcal{E}^*(Q,b)$ is a CP continuous linear map when $\mathcal{B}(\mathcal{H})$ is endowed with the weak*-topology.

Definition 1.2 *Let A be a unital non-type I nuclear C^*-algebra. Let ϕ be a pure state of A and γ an asymptotically inner endomorphism of A such that $\pi_\phi\gamma(A)'$ is a non-trivial abelian von Neumann algebra. Let $\mathcal{K} = \mathcal{K}(\mathcal{H})$ be as in the above definition and let U be a unitary in the multiplier algebra $M(\mathcal{K} \otimes A)$*[10] *of $\mathcal{K} \otimes A$. We call (A,ϕ,γ,U) a* measuring process *for $\mathcal{K}$.*

Proposition 1.3 *Let (A,ϕ,γ,U) be a measuring process and let $\mathcal{M} = \pi_\phi\gamma(A)'$. For each $Q \in \mathcal{M}$ and $\varphi \in \mathcal{K}^*$ define an $\mathcal{E}(Q,\varphi) \in \mathcal{K}^*$ by*

$$\mathcal{E}(Q,\varphi)(x) = \overline{\varphi \otimes \phi}(\mathrm{Ad}\,\bar{U}^*)(x \otimes Q), \quad x \in \mathcal{K},$$

where $\overline{\varphi \otimes \phi}$ is a unique extension of bounded functional $\varphi \otimes \phi\pi_\phi^{-1}$ of $\mathcal{K} \otimes \pi_\phi(A)$ to a weak*-continuous one of $(\mathcal{K} \otimes \pi_\phi(A))'' = \mathcal{B}(\mathcal{H} \otimes \mathcal{H}_\phi)$ and $\bar{U} = (\mathrm{id} \otimes \pi_\phi)(U)$. Then $\mathcal{E}$ is a CP instrument based on $\mathcal{M}$. We call $\mathcal{E}$ the CP instrument *obtained from (A,ϕ,γ,U).*

Note that we only use $\mathcal{M} = \pi_\phi\gamma(A)'$ for construction of the DL-instrument $\mathcal{E}$, not γ directly. In this sense the present scheme is not much different from the original one of von Neumann's on the technical level. But we hope that the present model makes a contribution for a clarification on the conceptual level.

When $\mathcal{E}_1$ and $\mathcal{E}_2$ are CP instruments based on $\mathcal{M}$ and (ξ_n) is a dense sequence in the unit sphere of $\mathcal{H}$ we define $d(\mathcal{E}_1,\mathcal{E}_2) \geq 0$ by

$$d(\mathcal{E}_1,\mathcal{E}_2) = \sum_n 2^{-n}\|\mathcal{E}_1(\,\cdot\,,\psi_n) - \mathcal{E}_2(\,\cdot\,,\psi_n)\|,$$

where ψ_n is the vector state of $\mathcal{K}$ defined by ξ_n. It follows that d is a distance on the set of DL-instruments based on $\mathcal{M}$. We can show the following:

[9] Since $\mathcal{M}$ is commutative this map is automatically CP.

[10] Identifying $\mathcal{K} \otimes A$ with $\mathcal{K} \otimes \pi_\phi(A)$ on $\mathcal{H} \otimes \mathcal{H}_\phi$, the multiplier algebra $M(\mathcal{K} \otimes A)$ is the set of $Q \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}_\phi)$ satisfying $Q(\mathcal{K} \otimes A), (\mathcal{K} \otimes A)Q \subset \mathcal{K} \otimes A$. For any unitary $U \in M(\mathcal{K} \otimes A)$ there is a unitary path $U_t,\ t \in [0,1]$ in $M(\mathcal{K} \otimes A)$ such that $U_0 = 1$, $U_1 = U$, and $t \mapsto xU_t, U_t x$ are continuous in $\mathcal{K} \otimes A$ (12.2.2 of [1]). Thus we may regard $\mathrm{Ad}\,U^*$ as representing a time development of $\mathcal{K} \otimes A$.

Proposition 1.4 *Let $\mathcal{M}$ be an abelian von Neumann algebra with separable predual. Then in the set of all CP instruments based on $\mathcal{M}$ is dense the set of CP instruments obtained from the measuring processes in the sense of Definition 1.2.*

We will sketch how to prove this. First of all we have to show that there is an asymptotically inner endomorphism γ and an irreducible representation π of some unital separable non-type I nuclear C*-algebra A such that $\mathcal{M} \cong \pi\gamma(A)'$. This is indeed possible for any unital separable non-type I nuclear C*-algebra, whose proof requires Glimm's result [5] (which shows UHF algebras are typical examples of non-type I C*-algebras), the existence result on endomorphisms [8] (for non-type I nuclear C*-algebras), and the following well-known statement on UHF algebras: For any such $\mathcal{M}$ as above there is a representation π such that $\pi(A)' \cong \mathcal{M}$, which will be shown in the same way as the examples of endomorphisms are given in Section 3. Thus we prepare (A, γ) and some irreducible representation π with $\pi\gamma(A)' \cong \mathcal{M}$.

Secondly by Ozawa's results (5.1-3 of [10]) all the CP instruments are realized by the measuring processes in his sense (stated in the beginning). For the proof we use the fact that $\mathcal{M} \times \mathcal{B}(\mathcal{H}) \ni (Q, b) \mapsto \mathcal{E}^*(Q, b) \in \mathcal{B}(\mathcal{H})$ is a completely positive, weak*-continuous bilinear map and express this map as the restriction of a *faithful* weak*-continuous representation of $\mathcal{M} \otimes \mathcal{B}(\mathcal{H})$ (by extending if necessary the representation obtained by Stinespring's procedure as in the proof of 4.2 of [10]). Namely for a CP instrument $\mathcal{E}$ based on $\mathcal{M}$ one finds a separable Hilbert space $\mathcal{L}$, a pure state ϕ of $\mathcal{K}(\mathcal{L})$, a normal unital representation ρ of $\mathcal{M}$ on $\mathcal{L}$, and a unitary U on $\mathcal{H} \otimes \mathcal{L}$ such that

$$\mathcal{E}(Q, \varphi)(x) = \overline{\varphi \otimes \phi}(\mathrm{Ad}\, U^*(x \otimes \rho(Q))), \quad Q \in \mathcal{M}, \ x \in \mathcal{K}.$$

We may assume that $\rho(\mathcal{M}) \cap \mathcal{K}(\mathcal{L}) = \{0\}$ by tensoring $\mathcal{L}$ by another infinite-dimensional separable Hilbert space if necessary and making obvious arrangements. Then we outfit an irreducible representation π of A on $\mathcal{L}$ such that $\pi\gamma(A)' = \rho(\mathcal{M})$.

Since this is done independently of U, we cannot expect that $U \in M(\mathcal{K} \otimes \pi(A))$. But, noting that $(\mathcal{K} \otimes \pi(A))'' = \mathcal{B}(\mathcal{H} \otimes \mathcal{L})$, Kadison's transitivity ([6] or 1.21.16 of [15]) tells us that one can find a unitary $u \in \mathcal{K} \otimes \pi(A) + \mathbb{C}1$ which equals U on any given finite-dimensional subspace.[11] Thus we can replace U by a unitary in $M(\mathcal{K} \otimes A)$ so that the resulting CP instrument is arbitrarily close to $\mathcal{E}$.

In the next section we will give an example of measuring process and explain the above definition of DL-instruments in more details. In section 3 we will show how to construct endomorphisms and irreducible representations in the case of UHF algebras of type k^∞. I wonder if this exposition gives some justification for γ being a magnifying glass of quantum effects.

This endeavor of mine was prompted by Professor M. Ozawa's lectures which struck home his success in placing Heisenberg's uncertainty principle in the right scheme involving inevitable

[11] Which shows a slightly stronger statement: For any CP instrument $\mathcal{E}$ and any finite number of pure states $\varphi_1, \ldots, \varphi_n$ on $\mathcal{K}$ there is a measuring process whose CP instrument is equal to $\mathcal{E}$ on $\varphi = \varphi_1, \ldots, \varphi_n$ (and any $Q \in \mathcal{M}$).

corrections on the principle and simultaneously bewildered me about the idea of measuring process itself, undoubtedly due to my ignorance, at the conference held in February 2013 organized by Professor T. Teruya for operator algebraists. I want to express my thanks to both of them for this opportunity and to Reiko, my wife, who accompanied me for this trip to Kyoto and then an expedition to Furano in March where an inceptive idea to the present note was conceived on a trail, for her unfailing company and patience in listening to my gibberish. I also want to extend my thanks to Professor I. Ojima for providing me with some information I should have known.

2 The case $\pi\gamma(A)' \cong \ell^\infty(\mathbb{N})$

Let A be a unital separable non-type I nuclear simple C*-algebra and Let γ be an asymptotically inner endomorphism of A such that $\pi_\phi\gamma(A)'$ is commutative. The existence of such γ follows from Theorem 3.3 of [8].[12] Let U be a unitary in $M(\mathcal{K} \otimes A)$. We will describe how the system (A, ϕ, γ, U) works as a measuring apparatus for the observed quantum system $\mathcal{K}$.

Let φ be a state of $\mathcal{K}$. We denote by id the identity representation of $\mathcal{K} = \mathcal{K}(\mathcal{H})$ on $\mathcal{H}$, where φ extends to a normal state of $\mathcal{B}(\mathcal{H}) = \mathcal{K}(\mathcal{H})''$. Then through the interaction with (A, ϕ) the state $\varphi \otimes \phi$ of the combined system $\mathcal{K} \otimes A$ changes to $(\varphi \otimes \phi)\mathrm{Ad}\, U^*$, and then to $T(\varphi) = (\varphi \otimes \phi)\mathrm{Ad}\, U^*(\mathrm{id} \otimes \gamma)$. Let $\pi_0 = (\mathrm{id} \otimes \pi_\phi)\mathrm{Ad}\, U^*(\mathrm{id} \otimes \gamma)$, which is a representation of $\mathcal{K} \otimes A$ on the Hilbert space $\mathcal{H} \otimes \mathcal{H}_\phi$. Then the commutant $\pi_0(\mathcal{K} \otimes A)'$ is equal to $\mathrm{Ad}\, \bar{U}^*(\mathbb{C}1 \otimes \pi_\phi\gamma(A)')$, where $\bar{U} = \mathrm{id} \otimes \pi_\phi(U)$. Note that $\pi_0(\mathcal{K} \otimes A)' = \pi_0(\mathcal{K} \otimes A)' \cap \pi_0(\mathcal{K} \otimes A)''$, the center of $\pi_0(\mathcal{K} \otimes A)''$.

Suppose that $\pi_\phi\gamma(A)' \cong L^\infty(\mathbb{N}, \mu)$, i.e., it is generated by minimal projections $E_1, E_2, \ldots$ on $\mathcal{H}_\phi$. Since $x \mapsto \pi_\phi\gamma(x)E_i$ is an irreducible representation of A on $E_i\mathcal{H}_\phi$, E_i is of infinite rank. Let $F_i = \mathrm{Ad}\, \bar{U}^*(1 \otimes E_i)$, which is a minimal projection of the center of $\pi_0(\mathcal{K} \otimes A)''$. If $\overline{\varphi \otimes \phi}$ denotes the natural extension to a normal state of $\mathcal{B}(\mathcal{H} \otimes \mathcal{H}_\phi)$ then

$$T(\varphi) = \sum_{i=1}^{\infty} \overline{\varphi \otimes \phi}(F_i\pi_0(\,\cdot\,)).$$

Since F_i is a minimal projection in $\pi_0(\mathcal{K} \otimes A)'$, the state $\omega_i = \overline{\varphi \otimes \phi}(F_i\pi_0(\,\cdot\,))/\overline{\varphi \otimes \phi}(F_i)$ is a pure state of $\mathcal{K} \otimes A$ for $\overline{\varphi \otimes \phi}(F_i) > 0$. Since F_i's are mutually orthogonal central projections, ω_i's are mutually disjoint.[13] Hence $T(\varphi)$ is the sum of phases with weights and Nature will pick up one according to the probability specified by $(\overline{\varphi \otimes \phi}(F_i))$.

[12]For example let $\nu_1, \nu_2, \ldots,$ be a sequence of irreducible representations of A such that all ν_n are mutually disjoint. If ρ is the direct sum $\nu_1 \oplus \nu_2 \oplus \cdots$ then the weak closure of $\rho(A)$ is equal to $\mathcal{B}(\mathcal{H}_1) \oplus \mathcal{B}(\mathcal{H}_2) \oplus \cdots$, where $\mathcal{H}_n$ is the representation space of ν_n. Then by Theorem 3.3 of [8] it follows that there is an asymptotically inner endomorphism γ of A such that $\pi\gamma$ is unitarily equivalent to ρ, which implies that $\pi\gamma(A)'$ is isomorphic to $\mathbb{C} \oplus \mathbb{C} \oplus \cdots$. The condition $u_0 = 1$ for the choice of u_t is not explicitly mentioned but follows from the proof. We could impose a finite number of conditions on γ of similar nature $\pi_i\gamma \cong \rho_i$ with mutually disjoint irreducible π_i and arbitrary ρ_i.

[13]ω_1 and ω_2, states of $B = \mathcal{K} \otimes A$, are disjoint if and only if there is a central sequence (x_n) in B such

Since $\varphi \mapsto \overline{\varphi\otimes\phi}(F_i)$ extends to a continuous positive linear map from $\mathcal{K}^*$ into $\mathbb{C}$ there is a positive operator P_i in $\mathcal{B}(\mathcal{H}) = \mathcal{K}(\mathcal{H})^{**}$ such that $\varphi(P_i) = \overline{\varphi\otimes\phi}(F_i)$. Since $\sum_i F_i = 1$ it follows that $\sum_i P_i = 1$. Note that restriction of $\overline{\varphi\otimes\phi}(F_i\pi_0(\,\cdot\,))$ to $\mathcal{K}$ is $\mathcal{E}(E_i,\varphi) = \overline{\varphi\otimes\phi}\mathrm{Ad}\,\bar{U}^*(\,\cdot\,\otimes E_i)$ and $\varphi(P_i) = \mathcal{E}(E_i,\varphi)(1)$ using the notation given in Definition 1.2.

Hence it follows that

$$T(\varphi)|\mathcal{K} = \sum_i \varphi(P_i)\frac{\mathcal{E}(E_i,\varphi)}{\varphi(P_i)},$$

where the sum is over i with $\varphi(P_i) > 0$ and $\varphi_i = \mathcal{E}(E_i,\varphi)/\varphi(P_i)$ is a state of $\mathcal{K}$, not necessarily a pure state. Here is our conclusion: After applying this measuring process to $\mathcal{K}$ Nature will transform φ to φ_i with probability $\varphi(P_i)$ for each $i = 1, 2, \ldots$.

Note that if $U = 1$ then $P_i = \overline{\varphi\otimes\phi}(E_i)1$ is independent of φ. Suppose that ϕ is given as a vector state by a unit vector $\psi_1 \in E_1\mathcal{H}_\phi$. If $U = 1$ then $T(\varphi) = \varphi\otimes\phi\gamma$ is pure and $P_1 = 1$; no information is gained. We choose a unit vector $\psi_i \in E_i\mathcal{H}_\phi$ for each $i > 1$ and choose a unitary $u_i \in A$ (or $A + \mathbb{C}1$ if A is non-unital) for $i \geq 1$ such that $\pi_\phi(u_i)\psi_1 = \psi_i$. The existence of such u_i follows from Kadison's transitivity [6] since π_ϕ is irreducible. We set $U = \sum_i e_{ii}\otimes u_i$; the summation converges to a unitary as a multiplier of $\mathcal{K}\otimes A$, where (e_{ij}) are matrix units generating $\mathcal{K}$. Since $\bar{U}\xi_i\otimes\psi_1 = \xi_i\otimes\psi_i$ where (ξ_i) is an orthnormal basis of $\mathcal{H}$ with $e_{ii}\xi_i = \xi_i$, it follows that $\overline{\varphi\otimes\phi}(F_i) = \overline{\varphi\otimes\phi}(\bar{U}^*(1\otimes E_i)\bar{U}) = \varphi(e_i)$ and $\varphi_i(x) = \mathrm{Tr}(e_{ii}x)$ for $x \in \mathcal{K}$ (when $\varphi(e_{ii}) > 0$). Hence for this choice of ϕ and U we obtain

$$T(\varphi)|\mathcal{K} = \sum_i \varphi(e_{ii})\mathrm{Tr}(e_{ii}\,\cdot\,),$$

which is what we expect by measuring e.g., the unbounded observable $\sum_n ne_{nn} \in M(\mathcal{K})$.

We should note that the von Neumann algebra $\mathcal{M}$ generated by all E_i plays the same role as the von Neumann algebra generated by M for the measuring process $(\mathcal{L},\phi,M,U)$ with $\mathcal{L} = \mathcal{H}_\phi$ we discussed in the beginning. Previously M is just an arbitrary self-adjoint operator on $\mathcal{H}_\phi$ and so the von Neumann algebra generated by M can contain a non-zero compact operator. But the present $\mathcal{M}$ must satisfy $\mathcal{M}\cap\mathcal{K}(\mathcal{H}_\phi) = \{0\}$.[14]

3 Endomorphisms

As we have noted, Theorem 3.3 of [8] serves to produce the desired endomorphisms for any separable non-type I nuclear simple C*-algebra. Here we show a concrete way to construct an asymptotically inner endomorphism γ and an irreducible representation π

that $\omega_1(x_n) \to 1$ and $\omega_2(x_n) \to 0$, which implies that $(\pi_{\omega_1}\oplus\pi_{\omega_2})(x_n) \to 1\oplus 0$ in the weak operator topology. (x_n) is a central sequence if it is bounded and $[x_n, y] \to 0$ for any $y \in B$. The C*-algebra consisting of central sequences (divided by some trivial ones) is considered to be the classical observables associated with B. We expect they reduce to numbers in a phase.

[14]But this is not important as it attained by tensoring an infinite-dimensional Hilbert space.

for the UHF algebra A of type k^∞ with $k>1$ such that $\pi\gamma(A)'$ is isomorphic to $\mathbb{C}^k$ but $A\cap\gamma(A)'=\mathbb{C}1$.[15]

We denote by M_k the C*-algebra of $k\times k$ matrices and denote by v the diagonal matrix $1\oplus\omega\oplus\omega^2\oplus\cdots\oplus\omega^{k-1}$ with $\omega=e^{i2\pi/k}$. We define an automorphism σ on $A=M_k\otimes M_k\otimes\cdots$ by

$$\sigma=\bigotimes_{n=1}^{\infty}\operatorname{Ad}v.$$

Since $v^k=1$ it follows that $\sigma^k=\mathrm{id}$. The fixed point algebra $A^\sigma=\{x\in A\mid\sigma(x)=x\}$ is isomorphic to A (see [16] and [7]). This is easy to see if you know of AF algebras and associated Bratteli diagrams [2]. We regard $M_k^{\otimes n}$ as the C*-subalgebra of A generated by the first n copies of M_k. Since $v^{\otimes n}=\sum_{j=0}^{k-1}\omega^jE_j\in M_k^{\otimes n}$, where E_j's are mutually orthogonal projections of $M_k^{\otimes n}$ of rank k^{n-1}, it follows that $(M_k^{\otimes n})^\sigma=\bigoplus_{j=0}^{k-1}E_jM_k^{\otimes n}E_j$ with $E_jM_k^{\otimes n}E_j\cong M_k^{\otimes(n-1)}$. Thus we can embed $M_k^{\otimes(n-1)}$ into $(M_k^{\otimes n})^\sigma$. We construct such embeddings consistently from $M_k\subset M_k^{\otimes 2}\subset M_k^{\otimes 3}\subset\cdots$ into $(M_k^{\otimes 2})^\sigma\subset(M_k^{\otimes 3})^\sigma\subset(M_k^{\otimes 4})^\sigma\subset\cdots$ preserving each level, where the closure of the union of the former (resp. latter) sequence is A (resp. A^σ); thus we obtain the isomorphism γ of A onto A^σ, which is the endomorphism we aimed at and is asymptotically inner as all the unital endomorphisms of A are. In our case this is also easy to see. Since $\gamma(M_k)\subset M_k^{\otimes 2}$, there is a (continuous) unitary path $u_t^{(1)},\ t\in[0,1]$ in $M_k^{\otimes 2}$ such that $u_0^{(1)}=1$ and $\operatorname{Ad}u_1^{(1)}(M_k)=\gamma(M_k)$. Since $\operatorname{Ad}u_1^{(1)}(M_k^{\otimes 2})=\gamma(M_k)\operatorname{Ad}u_1^{(1)}(1\otimes M_k)\subset M_k^{\otimes 3}$, it follows that both $\operatorname{Ad}u_1^{(1)}(1\otimes M_k)$ and $\gamma(1\otimes M_k)$ are unital subalgebras of $M_k^{\otimes 3}\cap\gamma(M_k)'$. Hence there is a unitary path $u_t^{(2)},\ t\in[0,1]$ in $M_k^{\otimes 3}\cap\gamma(M_k)'\cong M_k\otimes M_k$ such that $u_0^{(2)}=1$ and $\operatorname{Ad}(u_1^{(2)}u_1^{(1)})(1\otimes M_k)=\gamma(1\otimes M_k)$. In this way we construct a unitary path $u_t^{(n)},\ t\in[0,1]$ in $M_k^{\otimes(n+1)}\cap\gamma(M_k^{(n-1)})'$ such that $u_0^{(n)}=1$ and $\operatorname{Ad}(u_1^{(n)}u_1^{(n-1)}\cdots u_1^{(1)})(1^{\otimes(n-1)}\otimes M_k)=\gamma(1^{\otimes(n-1)}\otimes M_k)$. Combining all these unitary paths $u_t^{(n)}$ into one continuous unitary path $u_t,\ t\in[0,\infty)$ in A it follows that $\gamma(x)=\lim_{t\to\infty}\operatorname{Ad}u_t(x),\ x\in A$. Thus γ is an asymptotically inner endomorphism such that $\gamma(A)=A^\sigma$.

Let ϕ_0 be the pure state of M_k defined by $\phi_0(x)=x_{11}$ for $x=(x_{ij})\in M_k$. Since $\phi_0(v)=1$ we have that $\phi_0\operatorname{Ad}v=\phi_0$. Let $\phi=\phi_0\otimes\phi_0\otimes\cdots$, which is a σ-invariant pure state of A. Define a unitary U on the GNS representation space $\mathcal{H}_\phi$ by

$$U\pi_\phi(x)\Omega_\phi=\pi_\phi\sigma(x)\Omega_\phi,\ x\in A.$$

Then $\operatorname{Ad}U\pi_\phi(x)=\pi_\phi\sigma(x),\ x\in A$ and $U^k=1$. We can conclude that $\pi_\phi\gamma(A)'=U''\cong\mathbb{C}^k$ and that $\phi|\gamma(A)$ is a pure state.

By using the above fact we can construct more general examples. Since the tensor product of an infinitely many copies of A is isomorphic to A, one obtains a unital endomorphism of A by $A\cong A\otimes A\otimes\cdots\to\gamma(A)\otimes\gamma(A)\otimes\cdots\subset A\otimes A\otimes\cdots\cong A$, where the

[15] Which I do not have a specific reason to require but consider as a condition for γ to be close to an automorphism.

middle map is defined by $\gamma\otimes\gamma\cdots$. We denote this unital endomorphism by γ^∞. Let ϕ^∞ be the pure state of $A\cong A\otimes A\otimes\cdots$ defined by $\phi\otimes\phi\otimes\cdots$. Since ϕ^∞ is invariant under the action $\sigma^\infty=\sigma\otimes\sigma\otimes\cdots$ of the compact group $G=\mathbb{Z}_k\times\mathbb{Z}_k\times\cdots$ on $A\cong A\otimes A\otimes\cdots$ and $\phi^\infty|\gamma^\infty(A)$ is pure, it follows that $\pi_{\phi^\infty}\gamma^\infty(A)'$ is isomorphic to $\ell^\infty(\hat{G})\cong\ell^\infty(\mathbb{N})$.

Define a unit vector $\xi\in\mathcal{H}_\phi$ by $\xi=n^{-1/2}\sum_{j=1}^k\pi_\phi(e_{j1}^{(1)})\Omega_\phi$ and a pure state ψ of A by $\psi(x)=\langle\xi,\pi_\phi(x)\xi\rangle$, where $(e_{ij}^{(1)})$ is the matrix units of $M_k\subset A$. Then ψ is not σ-invariant but σ-covariant (i.e., $\pi_\psi=\pi_\phi$ is σ-covariant). Let ψ^∞ is the state of $A\cong A\otimes A\otimes\cdots$ defined by $\psi\otimes\psi\otimes\cdots$. Then ψ is covariant under the action obtained by restricting σ^∞ to the discrete subgroup $\hat{G}=\bigcup_{n=1}^\infty\mathbb{Z}_k\times\mathbb{Z}_k\times\cdots\times\mathbb{Z}_k(n$ factors) of G. Since there are no $\sigma^\infty|\hat{G}$-invariant states associated with π_{ψ^∞} we can conclude that $\pi_{\psi^\infty}\gamma(A)'\cong L^\infty(G)$, which is completely non-atomic. (If $\pi_{\psi^\infty}\gamma(A)'$ has a minimal projection E then a unit vector in $E\mathcal{H}_{\psi^\infty}$ defines a $\sigma^\infty|\hat{G}$-invariant state of A, which must be σ^∞-invariant, leading us to a contradiction.)

Let ϕ_1 be the pure state of M_k defined by $\phi_1(x)=k^{-1}\sum_{i,j}x_{ij}$ and let $\chi=\phi_1\otimes\phi_1\otimes\cdots$ as a state of A. We define γ in the most natural way, i.e., regarding $M_k^{\otimes(n-1)}=M_k(M_k^{\otimes(n-2)})$ as a giant matrix algebra $M_{k^{(n-1)}}$ in the natural way we embed $M_k^{\otimes(n-1)}$ into $(M_k^{\otimes n})^\sigma$ componentwise. Then one can easily see that $\chi\gamma=\chi$. Since χ and $\chi|\gamma(A)$ are both pure states, it follows that $\pi_\chi\gamma(A)'=\mathbb{C}1$ which implies that $A\cap\gamma(A)'=\mathbb{C}1$. In the representation π_χ there is a unitary U such that $\mathrm{Ad}\,U\pi_\chi(x)=\pi_\chi\gamma(x),\ x\in A$, i.e., γ is at least implemented by a unitary in some representation like an automorphism.

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